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Journal of Computational and Applied Mathematics 167 (2004) 59–71

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

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Complex minimax programming under generalized convexity☆

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Received 15 May 2003; received in revised form 4 August 2003

Abstract

We establish the Kuhn–Tucker-type sufficient optimality conditions for complex minimax programming under generalized invex functions. Subsequently, we apply these optimality criteria to formulate two dual models. We also establish weak, strong and strict converse duality theorems.

Keywords: Minimax programming; Generalized invex function; Duality theorem

1. Introduction

In this paper, we consider the following complex minimax programming problem involving generalized nonconvex complex problem:

$$\text{(P)}\quad \text{Minimize}\quad f(\xi) = \sup_{\varsigma\in W} \operatorname{Re}\phi(\xi,\varsigma)$$

$$\text{subject to}\quad \xi \in S^0 = \{\xi \in \mathbb{C}^{2n} : -g(\xi) \in S\},$$

where $\xi = (z,\bar{z}), \varsigma = (\omega,\varpi)$ for $z \in \mathbb{C}^n, \omega \in \mathbb{C}^m, \phi(\cdot,\cdot) : \mathbb{C}^{2n} \times \mathbb{C}^{2m} \to \mathbb{C}$ is analytic with respect to ξ, W is a specified compact subset in $\mathbb{C}^{2m}$, S is a polyhedral cone in $\mathbb{C}^p$ and $g : \mathbb{C}^{2n} \to \mathbb{C}^p$ is analytic.

Mathematical programming in complex space originated from Levinson's discussion of linear programming problems [12]. Actually, complex programming problems are extended from the optimization theory for real vector spaces, and $\mathbb{C}^n$ is isometrically isomorphic to $\mathbb{R}^{2n}$ under the isomorphism

☆ This research is supported by the University Grants Commission of India through Grant No. F 8-33/2001 (SR-I).

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0377-0427/$ - see front matter
doi:10.1016/j.cam.2003.09.045

of $z = x + iy \to (x, y)$, and so a function of n complex variables can be regarded as a function of $2n$ real variables. In order to get a sufficient condition for the existence of an optimal solution, we require extra assumptions to function like convexity or invexity in the necessary optimality condition. But, a nonlinear analytic function $f(z)$ cannot have a convex/invex real part in our requirements, see [1,7]. Thus in our investigation of sufficiency, we require analytic functions of the form $f(\xi)$ with $\xi = (z, \bar{z})$, that is, $f(z, \bar{z})$.

Several authors have recently been interested in the optimality conditions and the duality theorems for complex nonlinear programming. For details, one can consult [10,11,13–17,21–25], and the books of Craven [5] and Stancu-Minasian [24, Chapter 7]. Complex programs could be applied to electrical networks with alternating current with $z \in \mathbb{C}^n$ representing the current or voltage for an element of network. It is also employed to variant fields in electrical engineering like blind deconvolution, blind equalization, minimal entropy, maximum kurtosis, optimal receiver, etc. see [10,11] and references cited in there.

In this paper, we have answered partially a question raised in [14]. More precisely, we have established sufficient optimality condition for the problem considered in [14] under invexity and generalized invexity conditions. Furthermore, we have obtained duality results for the dual models considered in [14] under the aforesaid conditions.

2. Notations and preliminaries

Let $\mathbb{C}^n$ (or $\mathbb{R}^n$) denote an n-dimensional complex (or real) space, $\mathbb{C}^{m\times n}$ (or $\mathbb{R}^{m\times n}$) the collection of $m \times n$ complex matrices (or real matrices), $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n\colon x_i \geqslant 0, \text{ for all } i = 1, 2, \ldots, n\}$ the nonnegative orthant of $\mathbb{R}^n$, and $x \geqslant y$ represent $x - y \in \mathbb{R}^n_+$ for $x, y \in \mathbb{R}^n$. For $z \in \mathbb{C}^n$, let the real vectors $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ denote real and imaginary parts of each component of z, respectively, and write $\bar{z} = \mathrm{Re}(z) - \mathrm{i}\,\mathrm{Im}(z)$ as the conjugate of z. Given a matrix $A = [a_{ij}] \in \mathbb{C}^{m\times n}$, we use $\bar{A} = [\bar{a}_{ij}]$ to express its conjugate transpose. The inner product of $x, y \in \mathbb{C}^n$ is $\langle x, y\rangle = y^{\mathrm{H}} x$.

A nonempty subset S of $\mathbb{C}^m$ is said to be a *polyhedral cone* if there is an integer r and a matrix $K \in \mathbb{C}^{r\times m}$ such that $S = \{z \in \mathbb{C}^m\colon \mathrm{Re}(Kz) \geqslant 0\}$. The dual (also polar) of S is $S^* = \{\omega \in \mathbb{C}^m\colon z \in S \Rightarrow \mathrm{Re}\langle z, \omega\rangle \geqslant 0\}$. It is clear that $S = S^{**}$ if S is a polyhedral cone. Define the *manifold* $Q = \{\binom{\omega_1}{\omega_2} \in \mathbb{C}^{2n}\colon \omega_2 = \bar{\omega}_1\}$.

For $\xi = (z, \bar{z}) \in S^0$, we define $W(\xi) = \{\varsigma \in W\colon \mathrm{Re}\,\phi(\xi, \varsigma) = \sup_{\mu\in W} \mathrm{Re}\,\phi(\xi, \mu)\}$, and note that $W(\xi)$ is compact and nonempty.

For each $\varsigma \in W$, the functions $\phi(\cdot,\cdot) : \mathbb{C}^{2n} \times \mathbb{C}^{2m} \to \mathbb{C}$, and $g : \mathbb{C}^{2n} \to \mathbb{C}^p$ are differentiable with respect to $\xi = (z, \bar{z})$ if

$$\phi(z, \bar{z}; \varsigma) - \phi(z_0, \bar{z}_0; \varsigma) = \eta^{\mathrm{T}}(z, z_0)\nabla_z \phi(z_0, \bar{z}_0; \varsigma) + \eta^{\mathrm{H}}(z, z_0)\nabla_{\bar{z}} \phi(z_0, \bar{z}_0; \varsigma) + \mathrm{O}(|z - z_0|),$$

and

$$g(z, \bar{z}) - g(z_0, \bar{z}_0) = \eta^{\mathrm{T}}(z, z_0)\nabla_z g(z_0, \bar{z}_0) + \eta^{\mathrm{H}}(z, z_0)\nabla_{\bar{z}} g(z_0, \bar{z}_0) + \mathrm{O}(|z - z_0|),$$

where $\nabla_z \phi$, $\nabla_{\bar{z}} \phi$, $\nabla_z g$ and $\nabla_{\bar{z}} g$ denote, respectively, the vectors of partial derivatives of ϕ and g with respect to z and $\bar{z}$. Further $\mathrm{O}(|z - z_0|)/|z - z_0| \to 0$ as $z \to z_0$. Note that with $u \in \mathbb{C}^p$

$$\nabla_{\bar{z}} u^{\mathrm{H}} g(z, z_0) \equiv \nabla_z g(z_0, \bar{z}_0)\bar{u}.$$

We also need the following definitions, which are extensions of definitions given in [14,22,23].

Definition 2.1. (a) The real part of $\phi(\cdot,\varsigma)$ is said to be *invex* with respect to $\mathbb{R}_+$ on the manifold $Q \equiv \{(\omega_1,\omega_2)\in\mathbb{C}^{2n}: \omega_2 = \omega_1\}$ if there exists a function $\eta:\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}^n$ such that

$$\mathrm{Re}[\phi(z_2,\bar{z}_2;\varsigma)-\phi(z_1,\bar{z}_1;\varsigma)-\eta^{\mathrm{T}}(z_2,z_1)\nabla_z\phi(z_1,\bar{z}_1;\varsigma)-\eta^{\mathrm{H}}(z_2,z_1)\nabla_{\bar{z}}\phi(z_1,\bar{z}_1;\varsigma)]\geqslant 0$$

for all $z_1,z_2\in\mathbb{C}^n$.

The function $-g$ is said to be *invex* with respect to the polyhedral cone S if there exists a function $\eta:\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}^n$ such that

$$\mathrm{Re}\langle u, g(z_2,\bar{z}_2)-g(z_1,\bar{z}_1)-\eta^{\mathrm{T}}(z_2,z_1)\nabla_z g(z_1,\bar{z}_1)-\eta^{\mathrm{H}}(z_2,z_1)\nabla_{\bar{z}} g(z_1,\bar{z}_1)\rangle\geqslant 0$$

for all $z_1,z_2\in\mathbb{C}^n$.

In the above definition, if the strict inequality holds, the real part of $\phi(\cdot,\varsigma)$ and $-g$ are said to be *strict invex* with respect to $\mathbb{R}_+$ and the polyhedral cone S, respectively.

(b) The real part of $\phi(\cdot,\varsigma)$ is said to be *pseudoinvex* with respect to $\mathbb{R}_+$ on the manifold $Q\equiv \{(\omega_1,\omega_2)\in\mathbb{C}^{2n}: \omega_2=\omega_1\}$ if there exists a function $\eta:\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}^n$ such that

$$\mathrm{Re}[\eta^{\mathrm{T}}(z_2,z_1)\nabla_z\phi(z_1,\bar{z}_1;\varsigma)+\eta^{\mathrm{H}}(z_2,z_1)\nabla_{\bar{z}}\phi(z_1,\bar{z}_1;\varsigma)]\geqslant 0$$

$$\Rightarrow \mathrm{Re}[\phi(z_2,\bar{z}_2;\varsigma)-\phi(z_1,\bar{z}_1;\varsigma)]\geqslant 0 \quad \text{for all } z_1,z_2\in\mathbb{C}^n.$$

The function $-g$ is said to be *pseudoinvex* with respect to the polyhedral cone S if there exists a function $\eta:\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}^n$ such that

$$\mathrm{Re}\langle u,\eta^{\mathrm{T}}(z_2,z_1)\nabla_z g(z_1,\bar{z}_1)+\eta^{\mathrm{H}}(z_2,z_1)\nabla_{\bar{z}} g(z_1,\bar{z}_1)\rangle\geqslant 0$$

$$\Rightarrow \mathrm{Re}\langle u, g(z_2,\bar{z}_2)-g(z_1,\bar{z}_1)\rangle\geqslant 0, \quad \text{for all } z_1,z_2\in\mathbb{C}^n.$$

In the above definition, if the strict inequalities hold for all $z_2\neq z_1$, the real part of $\phi(\cdot,\varsigma)$ and $-g$ are said to be *strict pseudoinvex* with respect to η and $\mathbb{R}_+$ and the polyhedral cone S, respectively.

(c) The real part of $\phi(\cdot,\varsigma)$ is said to be *quasiinvex* with respect to $\mathbb{R}_+$ on the manifold $Q\equiv\{(\omega_1,\omega_2)\in\mathbb{C}^{2n}:\omega_2=\omega_1\}$ if there exists a function $\eta:\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}^n$ such that

$$\mathrm{Re}[\phi(z_2,\bar{z}_2;\varsigma)-\phi(z_1,\bar{z}_1;\varsigma)]\leqslant 0$$

$$\Rightarrow \mathrm{Re}[\eta^{\mathrm{T}}(z_2,z_1)\nabla_z\phi(z_1,\bar{z}_1;\varsigma)+\eta^{\mathrm{H}}(z_2,z_1)\nabla_{\bar{z}}\phi(z_1,\bar{z}_1;\varsigma)]\leqslant 0 \quad \text{for all } z_1,z_2\in\mathbb{C}^n.$$

The function $-g$ is said to be *quasiinvex* with respect to the polyhedral cone S if there exists a function $\eta:\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}^n$ such that

$$\mathrm{Re}\langle u, g(z_2,\bar{z}_2)-g(z_1,\bar{z}_1)\rangle\leqslant 0$$

$$\Rightarrow \mathrm{Re}\langle u,\eta^{\mathrm{T}}(z_2,z_1)\nabla_z g(z_1,\bar{z}_1)+\eta^{\mathrm{H}}(z_2,z_1)\nabla_{\bar{z}} g(z_1,\bar{z}_1)\rangle\leqslant 0 \quad \text{for all } z_1,z_2\in\mathbb{C}^n.$$

We shall use the following lemma for problem (P).

Lemma 2.1 (Liu [14]). *Let $\phi(\cdot,\cdot):\mathbb{C}^{2n}\times\mathbb{C}^{2m}\to\mathbb{C}$ be differentiable with respect to ξ for each $\varsigma\in W, g:\mathbb{C}^{2n}\to\mathbb{C}^p$ be differentiable with respect to ξ and let $S\subset\mathbb{C}^p$ be a polyhedral cone with nonempty interior. Let $\xi^0=(z_0,\bar{z}_0)$ be a solution to the minimax problem* (P). *Then there exist a* *positive integer s, scalars $\lambda_i \geqslant 0$, $i = 1,2,\dots,s$, $0 \neq u \in S^*$, and vectors $\varsigma_i \in W(\xi^0)$, $i = 1,2,\dots,s$, such that*

$$\sum_{i=1}^{s} \lambda_i \overline{\nabla_z \phi(\xi^0, \varsigma_i)} + \sum_{i=1}^{s} \lambda_i \nabla_{\bar{z}} \phi(\xi^0, \varsigma_i) + u^{\mathrm{T}} \overline{\nabla_z g(\xi^0)} + u^{\mathrm{H}} \nabla_{\bar{z}} g(\xi^0), \tag{2.1}$$
$$\mathrm{Re}\langle u, g(\xi^0)\rangle = 0.$$

Lemma 2.2 (Liu [14]; Necessary optimality conditions). *Let $\xi^0 = (z_0, \bar{z}_0)$ be an optimal solution of* (P) *and let $\phi(\cdot,\cdot) : \mathbb{C}^{2n} \times \mathbb{C}^{2m} \to \mathbb{C}$ be differentiable with respect to ξ for each $\varsigma \in W$, $g : \mathbb{C}^{2n} \to \mathbb{C}^p$ be differentiable with respect to ξ and let $S \subset \mathbb{C}^p$ be a polyhedral cone with nonempty interior. In addition, we suppose that the following conditions* (CQ) *holds*:

$$\text{(CQ)} \quad u^{\mathrm{T}} \overline{\nabla_z g(\xi^0)} + u^{\mathrm{H}} \nabla_{\bar{z}} g(\xi^0) = 0 \ \text{ implies } u = 0 \ \text{ for all } u \in \mathbb{C}^p. \tag{2.2}$$

Then there exist a positive integer s, scalars $\lambda_i \geqslant 0$, $i = 1,2,\dots,s$, $0 \neq u \in S^*$, and vectors $\varsigma_i \in W(\xi^0)$, $i = 1,2,\dots,s$, such that relations (2.1) *and* (2.2) *hold and*

$$\sum_{i=1}^{s} \lambda_i = 1. \tag{2.3}$$

3. Sufficient optimality conditions

In this section, we establish sufficient optimality criteria for problem (P) under generalized invex complex functions.

Theorem 3.2 (Sufficient optimality conditions). *Let $\xi^0 = (z_0, \bar{z}_0) \in S^0$ and assume that there exist a positive integer s, scalars $\lambda_i \geqslant 0$, $i = 1,2,\dots,s$, $0 \neq u \in S^*$, and vectors $\varsigma_i \in W(\xi^0)$, $i = 1,2,\dots,s$, satisfy conditions* (2.1)–(2.3). *If any one of following conditions holds*:

(a) *$\sum_{i=1}^{s} \lambda_i \phi(\cdot, \varsigma_i)$ has pseudoinvex real part with respect to η and $\mathbb{R}_+$ on the manifold Q and $g(\cdot)$ is a quasiinvex function with respect to the polyhedral cone $S \subset \mathbb{C}^p$ on the manifold Q.*

(b) *$\sum_{i=1}^{s} \lambda_i \phi(\cdot, \varsigma_i)$ has quasiinvex real part with respect to η and $\mathbb{R}_+$ on the manifold Q and $g(\cdot)$ is a strictly pseudoinvex function with respect to the polyhedral cone $S \subset \mathbb{C}^p$ on the manifold Q.*

(c) *$\sum_{i=1}^{s} \lambda_i \phi(\cdot, \varsigma_i) + u^{\mathrm{H}} g(\cdot)$ has pseudoinvex real part with respect to η and $\mathbb{R}_+$ on the manifold Q.*

Then $\xi^0 = (z_0, \bar{z}_0)$ is an optimal solution of (P).

Proof. Suppose on contrary that $\xi^0 = (z_0, \bar{z}_0)$ were not an optimal solution of (P). Then there exists a feasible solution $\xi = (z, \bar{z}) \in S^0$ such that

$$\sup_{\varsigma \in W} \mathrm{Re}\, \phi(\xi, \varsigma) < \sup_{\varsigma \in W} \mathrm{Re}\, \phi(\xi^0, \varsigma).$$

Since $\varsigma_i \in W(\xi^0)$, for all $i = 1,2,\dots,s$, we have

$$\mathrm{Re}\, \phi(\xi, \varsigma_i) < \mathrm{Re}\, \phi(\xi^0, \varsigma_i) \quad \text{for all } i = 1,2,\dots,s.$$

With $\lambda_i \geqslant 0,\ i = 1,2,\ldots,s$, and $\sum_{i=1}^{s} \lambda_i = 1$, we have

$$\operatorname{Re}\left[\sum_{i=1}^{s} \lambda_i \phi(\xi,\varsigma_i) - \sum_{i=1}^{s} \lambda_i \phi(\xi^0,\varsigma_i)\right] < 0. \tag{3.1}$$

Using the pseudoinvexity of $\sum_{i=1}^{s} \lambda_i \phi(\cdot,\varsigma_i)$, we get from inequality (3.1), we get

$$\operatorname{Re}\left\langle \eta^{\mathrm{T}}(z,z_0), \sum_{\imath=1}^{s} \lambda_\imath \overline{\nabla_z \phi(\xi^0,\varsigma_i)} + \sum_{\imath=1}^{s} \lambda_\imath \nabla_{\bar{z}} \phi(\xi^0,\varsigma_i) \right\rangle < 0. \tag{3.2}$$

Consequently, expressions (2.1) and (3.2) yield

$$\operatorname{Re}\langle \eta^{\mathrm{T}}(z,z_0), u^{\mathrm{T}}\overline{\nabla_z g(\xi^0)} + u^{\mathrm{H}} \nabla_{\bar{z}} g(\xi^0)\rangle > 0.$$

It follows that

$$\operatorname{Re}\langle u, \eta^{\mathrm{T}}(z,z_0)\nabla_z g(\xi^0) + \eta^{\mathrm{H}}(z,z_0)\nabla_{\bar{z}} g(\xi^0)\rangle > 0. \tag{3.3}$$

Utilizing the feasibility of ξ for (P), $u \in S^*$, and equality (2.2), we obtain

$$\operatorname{Re}\langle u, g(\xi)\rangle \leqslant 0 = \operatorname{Re}\langle u, g(\xi^0)\rangle. \tag{3.4}$$

Using the quasiinvexity of g, we get from inequality (3.4)

$$\operatorname{Re}\langle u, \eta^{\mathrm{T}}(z,z_0)\nabla_z g(\xi^0) + \eta^{\mathrm{H}}(z,z_0)\nabla_{\bar{z}} g(\xi^0)\rangle \leqslant 0,$$

which contradicts inequality (3.3). Therefore, $\xi^0 \in S^0$ is an optimal solution of (P).

Hypothesis (b) follows along the same lines as in (a).

If hypothesis (c) holds, from inequalities (3.1) and (3.4), we get

$$\operatorname{Re}\left[\sum_{i=1}^{s} \lambda_i \phi(\xi,\varsigma_i) + u^{\mathrm{H}} g(\xi)\right] < \operatorname{Re}\left[\sum_{i=1}^{s} \lambda_i \phi(\xi^0,\varsigma_i) + u^{\mathrm{H}} g(\xi^0)\right]. \tag{3.5}$$

Using the pseudoinvexity of $\sum_{i=1}^{s} \lambda_i \phi(\cdot,\varsigma_i) + u^{\mathrm{H}} g(\cdot)$ and (3.5), we get

$$\operatorname{Re}\left\langle \eta^{\mathrm{T}}(z,z_0), \sum_{i=1}^{s} \lambda_i \overline{\nabla_z \phi(\xi^0,\varsigma_i)} + \sum_{i=1}^{s} \lambda_i \nabla_{\bar{z}} \phi(\xi^0,\varsigma_i) + u^{\mathrm{H}}\overline{\nabla_z g(\xi^0)} + u^{\mathrm{H}} \nabla_{\bar{z}} g(\xi^0) \right\rangle < 0,$$

which contradicts equality (2.1). Therefore, $\xi^0 \in S^0$ is an optimal solution of (P). $\square$

4. The first dual model

From this section onwards, for $\xi = (z_1, \bar{z}_1) \in \mathbb{C}^{2n}$, we let

$$Y(\xi) = \left\{ (s,\lambda,\upsilon) \in \mathbb{N} \times \mathbb{R}_+^s \times \mathbb{C}^{2ms} \colon \lambda = (\lambda_1,\lambda_2,\ldots,\lambda_s) \in \mathbb{R}_+^s \text{ with } \sum_{i=1}^{s} \lambda_i = 1, \text{ and} \right.$$
$$\left. \upsilon = (\upsilon_1,\upsilon_2,\ldots,\upsilon_s) \text{ with } \upsilon_i \in W(\xi),\ i = 1,2,\ldots,s \right\}.$$

By the optimality conditions of the preceding section, we will show that the following formulation is a dual problem to the minimax complex problem:

$$\text{(DI)} \quad \max_{(s,\lambda,\varsigma)\in Y(\xi)} \sup_{(\xi,u,\bar{u},t)\in X(s,\lambda,\varsigma)} t,$$

where $X(s,\lambda,\varsigma)$ denotes the set of all $(\xi,u.\bar{u},t)\in\mathbb{C}^{2n}\times\mathbb{C}^p\times\mathbb{C}^p\times\mathbb{R}$ to satisfy

$$\sum_{i=1}^{s}\lambda_i\overline{\nabla_z\phi(\xi,\varsigma_i)}+\sum_{i=1}^{s}\lambda_i\nabla_{\bar{z}}\phi(\xi,\varsigma_i)+u^{\mathrm{T}}\overline{\nabla_z g(\xi)}+u^{\mathrm{H}}\nabla_{\bar{z}}g(\xi)=0, \tag{4.1}$$

$$\sum_{i=1}^{s}\lambda_i[\mathrm{Re}\phi(\xi,\varsigma_i)-t]\geqslant 0, \tag{4.2}$$

$$\mathrm{Re}\langle u,g(\xi)\rangle\geqslant 0, \tag{4.3}$$

$$(s,\lambda,\varsigma)\in Y(\xi), \tag{4.4}$$

$$0\neq u\in S^*. \tag{4.5}$$

We define the supremum over $X(s,\lambda,\varsigma)$ to be $-\infty$ if for a triplet $(s,\lambda,\varsigma)\in Y(\xi)$ the set $X(s,\lambda,\varsigma)=\phi$.

Then, we can derive the following weak duality theorem for (P) and (DI).

Theorem 4.1 (Weak duality). *Let $\xi=(z,\bar{z})\in S^0$ be a feasible solution of* (P) *and $(s,\lambda,\varsigma,\xi,u,\bar{u},t)$ be a feasible solution of* (DI). *If any one of the following holds*:

(a) $\sum_{i=1}^{s}\lambda_i\phi(\cdot,\varsigma_i)$ *has pseudoinvex real part with respect to η and $\mathbb{R}_+$ on the manifold Q and $g(\cdot)$ is a quasiinvex function with respect to the polyhedral cone $S\subset\mathbb{C}^p$ on the manifold Q.*

(b) $\sum_{i=1}^{s}\lambda_i\phi(\cdot,\varsigma_i)$ *has quasiinvex real part with respect to η and $\mathbb{R}_+$ on the manifold Q and $g(\cdot)$ is a strictly pseudoinvex function with respect to the polyhedral cone $S\subset\mathbb{C}^p$ on the manifold Q.*

(c) $\sum_{i=1}^{s}\lambda_i\phi(\cdot,\varsigma_i)+u^{\mathrm{H}}g(\cdot)$ *has pseudoinvex real part with respect to η and $\mathbb{R}_+$ on the manifold Q.*

Then $\sup_{\varsigma\in W}\mathrm{Re}\,\phi(\xi,\varsigma)\geqslant t$.

Proof. Suppose on contrary that

$$\sup_{\varsigma\in W}\mathrm{Re}\,\phi(\xi,\varsigma)<t.$$

Then, we have

$$\mathrm{Re}\,\phi(\xi,\varsigma)<t \quad \text{for all } \varsigma\in W.$$

It follows that

$$\mathrm{Re}[\lambda_i\phi(\xi,\varsigma_i)]\leqslant\lambda_i t \quad \text{for all } i=1,2,\ldots,s \tag{4.6}$$

with at least one strict inequality since $\lambda\neq 0$.

From inequalities (4.2) and (4.6), we have

$$\sum_{i=1}^{s} \mathrm{Re}[\lambda_i \phi(\xi, \varsigma_i)] < \sum_{i=1}^{s} \lambda_i t \leqslant \sum_{i=1}^{s} \mathrm{Re}[\lambda_i \phi(\xi, \varsigma_i)]. \tag{4.7}$$

If hypothesis (a) holds, using the pseudoinvexity of $\sum_{i=1}^{s} \lambda_i \phi(\cdot, \varsigma_i)$ and inequality (4.7), we get

$$\mathrm{Re}\left\langle \eta(z, z_1), \sum_{i=1}^{s} \lambda_i \overline{\nabla_z \phi(\xi, \varsigma_i)} + \sum_{i=1}^{s} \lambda_i \nabla_{\bar{z}} \phi(\xi, \varsigma_i) \right\rangle < 0. \tag{4.8}$$

From (4.1) and (4.8), we get

$$\mathrm{Re}\langle \eta(z, z_1), u^{\mathrm{T}} \overline{\nabla_z g(\xi)} + u^{\mathrm{H}} \nabla_{\bar{z}} g(\xi) \rangle > 0.$$

It follows that

$$\mathrm{Re}\langle u, \eta^{\mathrm{T}}(z, z_1) \nabla_z g(\xi) + \eta^{\mathrm{H}}(z, z_1) \nabla_{\bar{z}} g(\xi) \rangle > 0. \tag{4.9}$$

Utilizing the feasibility of ξ for (P), $u \in S^*$, and the inequality (4.3), we get

$$\mathrm{Re}\langle u, g(\varsigma) \rangle \leqslant 0 \leqslant \mathrm{Re}\langle u, g(\xi) \rangle. \tag{4.10}$$

Using the quasiinvexity of g and inequality (4.10), we get

$$\mathrm{Re}\langle u, \eta^{\mathrm{T}}(z, z_1) \nabla_z g(\xi) + \eta^{\mathrm{H}}(z, z_1) \nabla_{\bar{z}} g(\xi) \rangle \leqslant 0,$$

which contradicts inequality (4.9). Hence, the result holds.

Hypothesis (b) follows along with the same lines as (a).

If hypothesis (c) holds, from inequalities (4.7) and (4.10), we get

$$\mathrm{Re}\left[\sum_{i=1}^{s} \lambda_i \phi(\xi, \varsigma_i) + u^{\mathrm{H}} g(\xi)\right] < \mathrm{Re}\left[\sum_{i=1}^{s} \lambda_i \phi(\xi, \varsigma_i) + u^{\mathrm{H}} g(\xi)\right]. \tag{4.11}$$

Using the pseudoinvexity of $\sum_{i=1}^{s} \lambda_i \phi(\cdot, \varsigma_i) + u^{\mathrm{H}} g(\cdot)$ and (4.11), we get

$$\mathrm{Re}\left\langle \eta(z, z_1), \sum_{i=1}^{s} \lambda_i \overline{\nabla_z \phi(\xi, \varsigma_i)} + \sum_{i=1}^{s} \lambda_i \nabla_{\bar{z}} \phi(\xi, \varsigma_i) + u^{\mathrm{T}} \overline{\nabla_z g(\xi)} + u^{\mathrm{H}} \nabla_{\bar{z}} g(\xi) \right\rangle < 0,$$

which contradicts inequality (4.1). Hence the proof is complete. □

Theorem 4.2 (Strong duality). *Let ξ^0 be an optimal solution of problem* (P) *and condition* (CQ) *as defined in Lemma* 2.2 *is satisfied at ξ^0. Then there exist $(s, \lambda, \varsigma) \in Y(\xi^0)$ and $(\xi, u.\bar{u}, t) \in X(s, \lambda, \varsigma)$ such that $(s, \lambda, \varsigma, \xi^0, u, \bar{u}, t)$ is a feasible solution of* (DI). *If the hypothesis of Theorem* 4.1 *is also satisfied, then $(s, \lambda, \varsigma, \xi^0, u, \bar{u}, t)$ is an optimal solution of* (DI), *and the two problems* (P) *and* (DI) *have the same optimal value.*

Proof. Since ξ^0 is an optimal solution of (P) and condition (CQ) is satisfied, then Lemma 2.2 guarantees the existence of a positive s, scalars $\lambda_i \geqslant 0$, $i = 1,2,\dots,s$, $0 \neq u \in S^*$, and vectors $\varsigma_i \in W(\xi^0) = \{\varsigma \in W\colon \operatorname{Re}\phi(\xi^0,\varsigma) = \sup_{\mu\in W}\operatorname{Re}\phi(\xi^0,\mu)\}$, $i = 1,2,\dots,s$, such that

$$\sum_{i=1}^{s}\lambda_i\overline{\nabla_z\phi(\xi^0,\varsigma_i)} + \sum_{i=1}^{s}\lambda_i\nabla_{\bar z}\phi(\xi^0,\varsigma_i) + u^{\mathrm T}\overline{\nabla_z g(\xi^0)} + u^{\mathrm H}\nabla_{\bar z} g(\xi^0) = 0,$$

$$\operatorname{Re}\langle u, g(\xi^0)\rangle = 0$$

and $t = \operatorname{Re}\phi(\xi^0,\varsigma_i)$, $i = 1,2,\dots,s$. Thus $(s,\lambda,\varsigma,\xi^0,u,\bar u,t)$ is a feasible solution of (DI). The optimality of $(s,\lambda,\varsigma,\xi^0,u,\bar u,t)$ for (DI) follows from Theorem 4.1. □

Theorem 4.3 (Strict converse duality). *Let $\hat{\xi}$ and $(\hat s,\hat\lambda,\hat\varsigma,\hat{\xi},\hat u,\hat{\bar u},\hat t)$ be optimal solutions of* (P) *and* (DI), *respectively, and assume that the assumptions of Theorem* 4.2 *are fulfilled. If $\sum_{i=1}^{\hat s}\hat\lambda_i\phi(\cdot,\hat\varsigma_i)$ has strictly pseudoinvex real part with respect to η and $\mathbb{R}_+$ and g is quasiinvex with respect to the polyhedral cone S, then $\hat{\xi} = \hat\varsigma$, that is, $\hat\varsigma$ is an optimal solution of* (P).

Proof. We shall assume that $(\hat z,\hat{\bar z}) = \hat{\xi} \neq \hat\varsigma = (\hat z_1,\bar{\hat z}_1)$ and reach a contradiction. From Theorem 4.2, we know that

$$\sup_{v\in W}\operatorname{Re}\phi(\hat{\xi},v) = \hat t. \tag{4.12}$$

Utilizing the feasibility of $\hat{\xi}$ for (P), $\hat u \in S^*$, and inequality (4.3), we have

$$\operatorname{Re}\langle \hat u, g(\hat{\xi})\rangle \leqslant 0 \leqslant \operatorname{Re}\langle \hat u, g(\hat\varsigma)\rangle.$$

Using the quasiinvexity of g, we get from the above inequality

$$\operatorname{Re}\langle \hat u, \eta^{\mathrm T}(\hat z,\hat z_1)\nabla_z g(\hat{\xi}) + \eta^{\mathrm H}(\hat z,\hat z_1)\nabla_{\bar z} g(\hat{\xi})\rangle \leqslant 0. \tag{4.13}$$

From relations (4.1) and (4.13), we obtain

$$\operatorname{Re}\left\langle \eta(\hat z,\hat z_1), \sum_{i=1}^{\hat s}\hat\lambda_i\overline{\nabla_z\phi(\hat{\xi},\hat v_i)} + \sum_{i=1}^{\hat s}\hat\lambda_i\nabla_{\bar z}\phi(\hat{\xi},\hat v_i)\right\rangle \geqslant 0. \tag{4.14}$$

Using the strict pseudoinvexity of $\sum_{i=1}^{\hat s}\hat\lambda_i\phi(\cdot,\hat v_i)$, inequalities (4.14) and (4.2), we get

$$\sum_{i=1}^{\hat s}\operatorname{Re}[\hat\lambda_i\phi(\hat{\xi},\hat v_i)] > \sum_{i=1}^{\hat s}\operatorname{Re}[\hat\lambda_i\phi(\hat\varsigma,\hat v_i)] \geqslant \sum_{i=1}^{\hat s}\hat\lambda_i\hat t.$$

Therefore, there exists a certain i_0, such that

$$\operatorname{Re}\phi(\hat{\xi},\hat v_{i_0}) > \hat t.$$

It follows that

$$\sup_{v\in W}\operatorname{Re}\phi(\hat{\xi},v) \geqslant \operatorname{Re}\phi(\hat{\xi},\hat v_{i_0}) > \hat t,$$

which contradicts (4.12). Therefore, we conclude that $\hat{\xi} = \hat\varsigma$. Hence the proof is complete. □

5. The second dual model

We shall continue our discussion of duality model for (P) in this section by showing that the following problem (DII) is also a dual problem for (P).

$$\text{(DII)} \quad \max_{(s,\lambda,\varsigma)\in Y(\xi)} \sup_{(\xi,u,\bar{u},t)\in X(s,\lambda,\varsigma)} f(\xi),$$

where $X(s,\lambda,\varsigma)$ denotes the set of all $(\xi, u.\bar{u}) \in \mathbb{C}^{2n}\times\mathbb{C}^p\times\mathbb{C}^p$ to satisfy

$$\sum_{i=1}^{s}\lambda_i\overline{\nabla_z\phi(\xi,\varsigma_i)}+\sum_{i=1}^{s}\lambda_i\nabla_{\bar{z}}\phi(\xi,\varsigma_i)+u^{\mathrm{T}}\overline{\nabla_z g(\xi)}+u^{\mathrm{H}}\nabla_{\bar{z}}g(\xi)=0, \tag{5.1}$$

$$\mathrm{Re}\langle u, g(\xi)\rangle \geqslant 0, \tag{5.2}$$

$$f(\xi)=\sup_{v\in W}\mathrm{Re}\phi(\xi,v), \tag{5.3}$$

$$(s,\lambda,\varsigma)\in Y(\xi), \tag{5.4}$$

$$0\neq u\in S^*. \tag{5.5}$$

We define the supremum over $X(s,\lambda,\varsigma)$ to be $-\infty$ if for a triplet $(s,\lambda,\varsigma)\in Y(\xi)$ the set $X(s,\lambda,\varsigma)=\phi$.

We shall establish the following weak, strong and strict converse duality theorem for (P) and (DII).

Theorem 5.1 (Weak duality). *Let $\varsigma=(z,\bar{z})\in S^0$ be a feasible solution of* (P) *and $(s,\lambda,v,\xi,u,\bar{u},t)$ be a feasible solution of* (DII). *If any one of the following holds*:

(a) $\sum_{i=1}^{s}\lambda_i\phi(\cdot,v_i)$ *has pseudoinvex real part with respect to η and $\mathbb{R}_+$ on the manifold Q and $g(\cdot)$ is a quasi invex function with respect to the polyhedral cone $S\subset\mathbb{C}^p$ on the manifold Q.*

(b) $\sum_{i=1}^{s}\lambda_i\phi(\cdot,v_i)$ *has quasiinvex real part with respect to η and $\mathbb{R}_+$ on the manifold Q and $g(\cdot)$ is a strictly pseudoinvex function with respect to the polyhedral cone $S\subset\mathbb{C}^p$ on the manifold Q.*

(c) $\sum_{i=1}^{s}\lambda_i\phi(\cdot,v_i)+u^{\mathrm{H}}g(\cdot)$ *has pseudoinvex real part with respect to η and $\mathbb{R}_+$ on the manifold Q.*

Then $f(\varsigma)\geqslant f(\xi)$.

Proof. Suppose contrary to the result, we then have

$$f(\varsigma)<f(\xi);$$

that is,

$$\sup_{v\in W}\mathrm{Re}\,\phi(\varsigma,v)<\sup_{v\in W}\mathrm{Re}\,\phi(\xi,v).$$

Then, we have

$$\mathrm{Re}\,\phi(\varsigma,v)<\sup_{v\in W}\mathrm{Re}\phi(\xi,v)\quad\text{for all } v\in W. \tag{5.6}$$

Since $v_i \in W(\xi)$ for all $i = 1, 2, \ldots, s$, we obtain

$$\sup_{v \in W} \mathrm{Re}\phi(\xi, v) = \mathrm{Re}\phi(\xi, v_i) \quad \text{for all } i = 1, 2, \ldots, s. \tag{5.7}$$

From relations (5.6) and (5.7), we obtain

$$\mathrm{Re}\,\phi(\varsigma, v) < \mathrm{Re}\,\phi(\xi, v_i) \quad \text{for all } i = 1, 2, \ldots, s, v \in W.$$

It follows that

$$\mathrm{Re}[\lambda_i \phi(\varsigma, v_i)] \leqslant \mathrm{Re}[\lambda_i \phi(\xi, v_i)] \quad \text{for all } i = 1, 2, \ldots, s$$

with atleast one strict inequality since $\lambda \neq 0$.

Thus, we have

$$\sum_{i=1}^{s} \mathrm{Re}[\lambda_i \phi(\varsigma, v_i)] < \sum_{i=1}^{s} \mathrm{Re}[\lambda_i \phi(\xi, v_i)]. \tag{5.8}$$

Using the pseudoinvexity of the real part of $\sum_{i=1}^{s} \lambda_i \phi(\varsigma, v_i)$ and the above inequality, we get

$$\mathrm{Re}\left\langle \eta(z, z_1), \sum_{i=1}^{s} \lambda_i \overline{\nabla_z \phi(\xi, v_i)} + \sum_{i=1}^{s} \lambda_i \nabla_{\bar{z}} \phi(\xi, v_i) \right\rangle < 0. \tag{5.9}$$

From inequalities (5.1) and (5.9), we get

$$\mathrm{Re}\langle \eta(z, z_1), u^{\mathrm{T}} \overline{\nabla_z g(\xi)} + u^{\mathrm{H}} \nabla_{\bar{z}} g(\xi) \rangle > 0. \tag{5.10}$$

Thus, we have

$$\mathrm{Re}\langle u, \eta^{\mathrm{T}}(z, z_1) \nabla_z g(\xi) + \eta^{\mathrm{H}}(z, z_1) \nabla_{\bar{z}} g(\xi) \rangle > 0. \tag{5.11}$$

By the feasibility of ς for (P), $u \in S^*$, and inequality (5.2), we get

$$\mathrm{Re}\langle u, g(\varsigma) \rangle \leqslant 0 \leqslant \mathrm{Re}\langle u, g(\xi) \rangle. \tag{5.12}$$

Using the quasiinvexity of g and inequality (5.12), we get

$$\mathrm{Re}\langle u, \eta^{\mathrm{T}}(z, z_1) \nabla_z g(\xi) + \eta^{\mathrm{H}}(z, z_1) \nabla_{\bar{z}} g(\xi) \rangle \leqslant 0,$$

which contradicts inequality (5.11). Hence the result is true.

Hypothesis (b) follows along with the same lines as (a).

If hypothesis (c) holds, from inequalities (5.8) and (5.12), we get

$$\sum_{i=1}^{s} \mathrm{Re}[\lambda_i \phi(\varsigma, v_i) + u^{\mathrm{H}} g(\varsigma)] < \sum_{i=1}^{s} \mathrm{Re}[\lambda_i \phi(\xi, v_i) + u^{\mathrm{H}} g(\xi)]. \tag{5.13}$$

By the pseudoinvexity of $\sum_{i=1}^{s} \lambda_i \phi(\cdot, v_i) + u^{\mathrm{H}} g(\cdot)$ and the above inequality, we get

$$\mathrm{Re}\left\langle \eta(z, z_1), \sum_{i=1}^{s} \lambda_i \overline{\nabla_z \phi(\xi, v_i)} + \sum_{i=1}^{s} \lambda_i \nabla_{\bar{z}} \phi(\xi, v_i) + u^{\mathrm{T}} \overline{\nabla_z g(\xi)} + u^{\mathrm{H}} \nabla_{\bar{z}} g(\xi) \right\rangle < 0,$$

which contradicts inequality (5.1). Hence the result of theorem holds. □

Theorem 5.2 (Strong duality). *Let ς^0 be an optimal solution of problem* (P) *and the condition* (CQ) *as defined in Lemma* 2.2 *is satisfied at ς^0. Then there exist $(s, \lambda, v) \in Y(\varsigma^0)$ and $(\varsigma^0, u.\bar{u}) \in X(s, \lambda, v)$ such that $(s, \lambda, v, \varsigma^0, u, \bar{u})$ is a feasible solution of* (DII). *If the hypothesis of Theorem* 5.1 *is also satisfied, then $(s, \lambda, v, \varsigma^0, u, \bar{u})$ is an optimal solution of* (DII), *and the two problems* (P) *and* (DII) *have the same optimal value.*

Proof. By Lemma 2.2, there exist $(s, \lambda, v) \in Y(\varsigma^0)$ and $(\varsigma^0, u.\bar{u}) \in X(s, \lambda, v)$ such that $(s, \lambda, v, \varsigma^0, u, \bar{u})$ is a feasible solution of (DII). Since (P) and (DII) have the same objective function, the optimality of $(s, \lambda, v, \varsigma^0, u, \bar{u})$ for (DII) follows from Theorem 5.1. □

Theorem 5.3 (Strict converse duality). *Let $\hat{\varsigma}$ and $(\hat{s}, \hat{\lambda}, \hat{v}, \hat{\varsigma}, \hat{u}, \hat{\bar{u}})$ be optimal solutions of* (P) *and* (DII) *respectively, and assume that the assumptions of Theorem* 5.2 *are fulfilled. If $\sum_{i=1}^{\hat{s}} \hat{\lambda}_i \phi(\cdot, \hat{v}_i)$ has strictly pseudoinvex real part with respect to η and $\mathbb{R}_+$ and g is quasiinvex with respect to the polyhedral cone S, then $\hat{\varsigma} = \hat{\xi}$; that is, $\hat{\xi}$ is an optimal solution of* (P).

Proof. We shall assume that $(\hat{z}, \hat{\bar{z}}) = \hat{\varsigma} \neq \hat{\xi} = (\hat{z}_1, \hat{\bar{z}}_1)$ and reach a contradiction. From Theorem 5.2, we know that

$$\sup_{v \in W} \mathrm{Re}\, \phi(\hat{\varsigma}, v) = \sup_{v \in W} \mathrm{Re}\, \phi(\hat{\xi}, v). \tag{5.14}$$

Utilizing the feasibility of $\hat{\varsigma}$ for (P), $\hat{u} \in S^*$, and inequality (5.2), we have

$$\mathrm{Re}\langle \hat{u}, g(\hat{\varsigma}) \rangle \leqslant 0 \leqslant \mathrm{Re}\langle \hat{u}, g(\hat{\xi}) \rangle.$$

Using the quasiinvexity of g, we get from the above inequality

$$\mathrm{Re}\langle \hat{u}, \eta^{\mathrm{T}}(\hat{z}, \hat{z}_1) \nabla_z g(\hat{\xi}) + \eta^{\mathrm{H}}(\hat{z}, \hat{z}_1) \nabla_{\bar{z}} g(\hat{\xi}) \rangle \leqslant 0. \tag{5.15}$$

From relations (5.1) and (5.15), we obtain

$$\mathrm{Re}\left\langle \eta(\hat{z}, \hat{z}_1), \sum_{i=1}^{\hat{s}} \hat{\lambda}_i \overline{\nabla_z \phi(\hat{\xi}, \hat{v}_i)} + \sum_{i=1}^{\hat{s}} \hat{\lambda}_i \nabla_{\bar{z}} \phi(\hat{\xi}, \hat{v}_i) \right\rangle \geqslant 0. \tag{5.16}$$

Using the strict pseudoinvexity of $\sum_{i=1}^{\hat{s}} \hat{\lambda}_i \phi(\cdot, \hat{v}_i)$ and the above inequalities, we get

$$\sum_{i=1}^{\hat{s}} \mathrm{Re}[\hat{\lambda}_i \phi(\hat{\varsigma}, \hat{v}_i)] > \sum_{i=1}^{\hat{s}} \mathrm{Re}[\hat{\lambda}_i \phi(\hat{\xi}, \hat{v}_i)].$$

Therefore, there exists a certain i_0, such that

$$\mathrm{Re}\, \phi(\hat{\varsigma}, \hat{v}_{i_0}) > \mathrm{Re}\, \phi(\hat{\xi}, \hat{v}_{i_0}).$$

It follows that

$$\sup_{v\in W}\operatorname{Re}\phi(\hat{\varsigma},v)\geqslant\operatorname{Re}\phi(\hat{\varsigma},\hat{v}_{i_0})>\operatorname{Re}\phi(\hat{\xi},\hat{v}_{i_0})=\sup_{v\in W}\operatorname{Re}\phi(\hat{\xi},v),$$

which contradicts (5.14). Therefore, we conclude that $\hat{\xi}=\hat{\varsigma}$. Hence the proof is complete. □

6. Some further development

(1) Whether the results developed in this paper can hold for nondifferentiable complex minimax fractional problem (P*) involving generalized invex functions?

$$(\mathrm{P}^*)\quad \text{Minimize}\quad f(\varsigma)=\sup_{v\in W}\operatorname{Re}\frac{\phi(\varsigma,v)+(z^{\mathrm{H}}Bz)^{1/2}}{\psi(\varsigma,v)+(z^{\mathrm{H}}Az)^{1/2}}$$

$$\text{subject to}\quad \varsigma\in S^0=\{\varsigma\in\mathbb{C}^{2n}\colon -g(\varsigma\in S)\}.$$

(2) Can the objective and constraint functions in the complex minimax programming problem (P) be replaced by type I and generalized type I functions.

7. Uncited references

[2–4,6,8,9,18–20].

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